

Surface Roughness Effects for Newtonian and Non-Newtonian Lubricants

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ABSTRACT

In the present study, a numerical solution is presented to investigate the combined effects of lubricant rheology and surface roughness parameters on the characteristic of elastohydrodynamic lubrication problem for point contact. Sinusoidal waviness for surface roughness is considered to show the effect of changing amplitude and wavelength on the film thickness and pressure profile. The results show that, surface waviness leads to random fluctuations of pressure and film thickness for both modes of lubricant of Newtonian and non-Newtonian. As the amplitude of the waviness increase, the more fluctuations of pressure profile and film profile occur. Changing the wavelength of wavy surface shows that the amplitude of the pressure fluctuation becomes a little more remarkable as wavelength increase. The difference between Newtonian and non-Newtonian lubricant is that the non-parallel film shape in the contact region occurs for non-Newtonian model in addition to the reduction in film thickness compared to the Newtonian model.

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1. INTRODUCTION

Mechanical elements often operate under the regime of elastohydrodynamic lubrication, for example; contacts between gear teeth, cams and between the rolling elements. This type of lubrication has been studied extensively both theoretically and experimentally over the past decades. Due to the complexities encountered in the solution of an elastohydrodynamic lubrication problem, many assumptions to the numerical solution in the past have been done, such as Newtonian rheology, smooth surfaces,... etc. For a real system and for an accurate prediction of film thickness and pressure profile,

the effects of lubricant rheology and surface roughness should be considered in the numerical solution.

Several elastohydrodynamic lubrication studies employing different shear-thinning models such as the Ree-Eyring model, the power-law model, the Johnson-Tevaarwerk model and others have been developed. The Ree-Eyring model is the most widely used rheological model, as shown for example, by Hirst and Moore [1], Chang et al. [2], Kumar et al. [3] and Liu et al. [4,5]. The results of their numerical works showed that the minimum film thickness within the nominal contact zone for line and point contact

simulations does not change significantly from the Newtonian solution to a certain value of slide-roll ratio.

Surface roughness affects the formation of a lubricating oil film thickness and pressure performance as shown by many researchers. The effect of surface roughness should be considered in the analysis especially on thin film lubrication as surface roughness is comparable to the film thickness. Rough surface of elastohydrodynamic lubrication have been studied with two types of approaches. The first approach is the stochastic method as shown by many researchers such as Patir and Cheng [6], Epstein et al. [7], Wang et al. [8] and Masjedi and Khonsari [9]. The other approach is the deterministic method (see for example, Hu and Zhu [10], Jacod et al. [11], Yang et al. [12] and Wang et al. [13]).

The combined effects of lubricant rheology and surface roughness have been studied for example by Li et al. [14], Lin et al. [15] and Javorova et al. [16] for hydrodynamic lubrication. Li-Ming et al. [17] investigated the coupling effects of flow rheology of the power law and surface roughness on the elastohydrodynamic circular contact problems numerically using multilevel multi-integration algorithm by solving the average Reynolds type equation, elasticity deformation, viscosity and density pressure relations equations, and load balance equations simultaneously. Their results showed that the transverse type roughness enhance the pressure and film thickness in the central contact region. Ildiko' Ficza et al. [18] presented a mathematical model using multigrid method to solve the Reynolds equation for an isothermal elastohydrodynamic point contact with a single transverse ridge on one of the surfaces for Newtonian and non-Newtonian fluid based on Ree-Eyring model. Their results showed that the pressure and film thickness are closer to experimental observations when using non-Newtonian model. Kumar and Kumar [19] studied the effect of sinusoidal surface roughness on elastohydrodynamic lubrication characteristics for line contact problem for shear-thinning lubricants described by Carreau-Type Shear-Thinning Lubricants along with Doolittle-Tait equation for lubricant compressibility. Linlin and Jiajun [20] studied the effects of surface roughness on the

lubrication performances for point contact problem using the multi-grid technique. Kumar and Kumar [19] and Linlin and Jiajun [20] found that the presence of surface roughness can cause the random fluctuations of the pressure and film thickness and the fluctuations can become more dramatic for the rougher surfaces.

Until now, the study of the coupling effects of lubricant rheology and surface roughness for the elastohydrodynamic lubrication point contact problems is not complete. Therefore, the present analysis represents the numerical solution using Newton-Raphson technique with Gauss-Seidel iteration method to solve the Reynolds equation in order to investigate the effects of lubricant rheology using Ree-Eyring model and sinusoidal waviness for surface roughness on the elastohydrodynamic lubrication point contact problems. Effects of changing various parameters of waviness such as amplitude and wavelength and effect of using lubricant rheology such as Ree-Eyring model and Newtonian one are analyzed and discussed in relation to the film thickness and pressure characteristics.

2. BACKGROUND THEORY

The Eyring sinh law is used in the present analysis to describe the nonlinear viscous function of the lubricant. The constitutive law equation can be written as:

$$\gamma' = \frac{\tau_o}{\eta} \sinh\left(\frac{\tau_e}{\tau_o}\right) \quad (1)$$

Where, γ' is the shear rate, τ_o represent the Eyring shear stress and τ_e represent the equivalent shear stress.

The two-dimensional Reynolds equation for a general non-Newtonian fluid can be written in dimensionless form as:

$$\frac{\partial}{\partial X} \left(\frac{\bar{\rho} H^3}{\bar{\eta}} \frac{\partial P}{\partial X} \varphi_x \right) + \frac{1}{K^2} \frac{\partial}{\partial Y} \left(\frac{\bar{\rho} H^3}{\bar{\eta}} \frac{\partial P}{\partial Y} \varphi_y \right) = \lambda \frac{\partial}{\partial X} (\bar{\rho} H) \quad (2)$$

Where the following dimensionless variables apply:

$$X=x/b, Y=y/a, \bar{\eta} = \eta/\eta_o, \bar{\rho} = \frac{\rho}{\rho_o}, H=hR_x/b^2, P=p/p_{Her}$$

$$\text{and } \lambda = \frac{12\mu\eta_o R_x^2}{b^3 P_{Her}}$$

φ_x and φ_y are the effective viscosities (or flow factors as given by Greenwood [21]). For the

Eyring fluid the effective viscosities can be given as:

$$\varphi_x = \cosh\left(\frac{\tau_m}{\tau_o}\right), \varphi_y = \frac{\sinh(\tau_m/\tau_o)}{\tau_m/\tau_o} \quad \text{and}$$

$$\tau_m = \tau_o \sinh^{-1}[\eta u_s(t)/(\tau_o h)]$$

Where τ_m represent the mean shear stress for the Eyring model.

Figure 1 shows the geometrical representation of the elastohydrodynamic problem, where m and n is the inlet and outlet lengths respectively, while l is the lateral boundary.

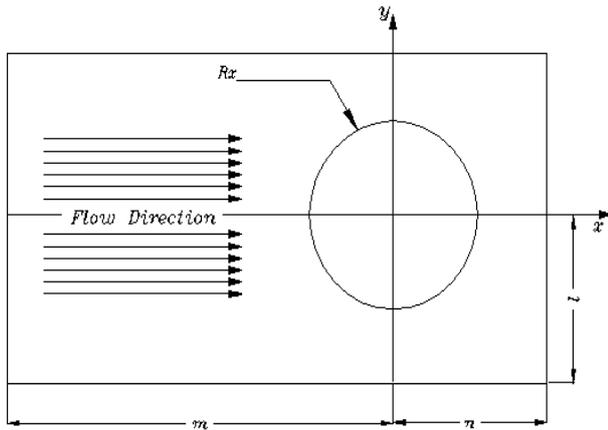


Fig. 1. Geometrical representation of the domain.

The dimensionless film thickness equation can be given as:

$$H(X,Y) = H_0 + \frac{(X-m)^2}{2} + \frac{K^2 R_x (Y-l)^2}{R_y} + \frac{R_x \delta(X,Y)}{b^2} + A \sin\left(\frac{2\pi X}{\lambda}\right) \sin\left(\frac{2\pi Y}{\lambda}\right) \quad (3)$$

Where δ is the total elastic deformation of the contiguous bodies in contact is given as shown by Al-Samieh and Rahnejat [22] and Al-Samieh [23]. The last term in film thickness equation (3) represent the assumed sinusoidal surface topography with A is the dimensionless amplitude and λ is the dimensionless wavelength.

The density variation with pressure is defined by Dowson and Higginson [24] as:

$$\bar{\rho} = 1 + \frac{\varepsilon P P_{Her}}{1 + \zeta P P_{Her}} \quad (4)$$

Where ε and ζ are constants dependent upon the type of lubricant used.

The viscosity relation with pressure is given by Roelands [25] as:

$$\bar{\eta} = \exp[\ln \eta_o + 9.67] [(1 + 5.1 * 10^{-9} P P_{Her})^Z - 1] \quad (5)$$

where:
$$Z = \frac{\alpha}{5.1 * 10^{-9} [\ln \eta_o + 9.67]}$$

The Reynolds' equation can be solved using Newton-Raphson technique in the following numerical form:

$$\sum_{l=2}^{my-1} \sum_{k=2}^{mx-1} J_{k,l}^{i,j} \Delta \bar{P}_{k,l} = -F_{i,j} \quad (6)$$

where, the Jacobian matrix is given in terms of the residual derivatives as:

$$J_{k,l}^{i,j} = \frac{\partial F_{i,j}}{\partial P_{k,l}}$$

Using the Gauss-Seidel iteration method, equation (6) can be written as:

$$\Delta P_{k,l}^n = (-F_{i,j} - J_{k-1,l}^{i,j} \Delta P_{k-1,l}^n - J_{k+1,l}^{i,j} \Delta P_{k+1,l}^{n-1} - J_{k,l-1}^{i,j} \Delta P_{k,l-1}^n - J_{k,l+1}^{i,j} \Delta P_{k,l+1}^{n+1}) / J_{k,l}^{i,j}$$

where n is the iteration counter.

The pressure can be updated according to:

$$P_{i,j}^n = P_{i,j}^{n-1} + \Omega \Delta P_{i,j}^n \quad (7)$$

where Ω is an under-relaxation factor, which ranges from 0.05 to 0.1.

The convergence criteria for the pressure and load balance equations are:

$$\left[\frac{\sum_i \sum_j (P_{i,j}^n - P_{i,j}^{n-1})^2}{MxN} \right]^{0.5} \leq 10^{-3} \quad (8)$$

$$\left| \iint P(X,Y) dXdY - \frac{2}{3} \pi \right| \leq 10^{-3}$$

where M and N are the total nodal points in both the X and Y directions.

3. RESULTS AND DISCUSSION

The geometry, material and lubricant properties are summarized in Table 1. Figure 2 shows the film thickness and pressure profile for smooth surfaces for Newtonian and non-Newtonian fluid using Ree-Eyring model for the operating conditions of dimensionless load of 3.913×10^{-7} , dimensionless speed of 3.287×10^{-11} and material parameter of 3067. Under these conditions, the minimum film thickness obtained numerically for Newtonian model is about 381 nm while that obtained using Hamrock and Dowson [26] formula is about 396 nm (the error is being less than 4 %). It can be seen that, the non-Newtonian model affects the

film thickness and pressure spike, where the film profile through the contact region is not parallel as in case of Newtonian case and the film thickness value is smaller compared to that formed using Newtonian lubricant and the pressure spike disappear.

Table 1. Geometry, material, lubricant properties and operating conditions.

Viscosity η_0	0.28 Pa.s	Eyring model shear stress τ_0	8×10^6 Pa
Pressure of viscosity coefficient α	24×10^{-9} Pa ⁻¹	Reduced modulus of elasticity E	123.8×10^9 Pa
ϵ	5.83×10^{-10} Pa	Load w	20 N
ξ	1.68×10^{-9} Pa	Speed u	0.3 m/s
Radius of ball R_x	0.02 m		

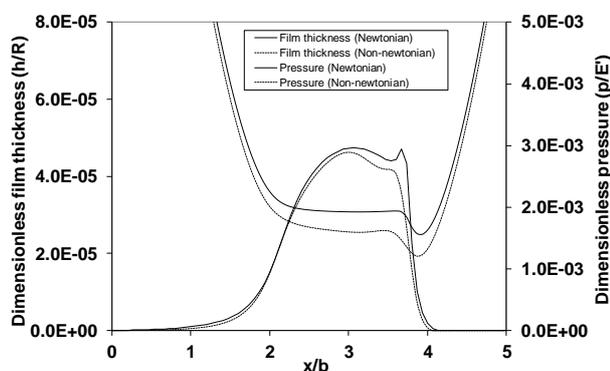
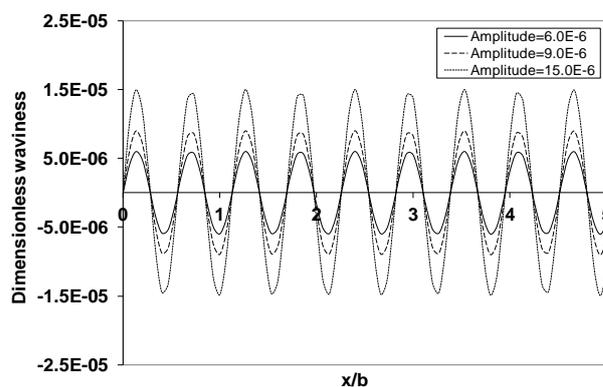


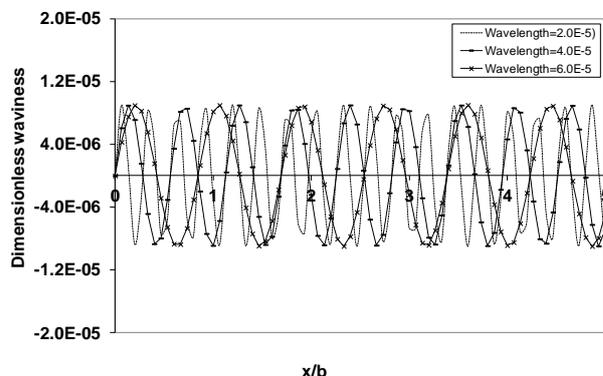
Fig. 2. Dimensionless film thickness and pressure profile for Newtonian and non-Newtonian models.

The next step in this paper is to study the rough surface elastohydrodynamic lubrication problems for Newtonian and non-Newtonian models. In this case a sinusoidal wavy surface is considered to show the effects of variation of amplitude and wavelength of the waviness on pressure and film profile. Figures 3a and 3b shows the dimensionless waviness patterns used in the current analysis, the first one represents the dimensionless amplitude variation while the second one is for dimensionless wavelength variation. Comparisons of the film thickness and pressure distribution under the different surface amplitude waviness (variation of amplitude in dimensionless form from 6×10^{-6} to 15×10^{-6}) with smooth surface for the same working conditions mentioned previously are presented in Figs. 4a and 4b at dimensionless wavelength of 4×10^{-5} for a Newtonian case. It is clear that, variation of amplitude surface waviness causes fluctuation on pressure and film thickness within the nominal

contact zone and as the amplitude of surface waviness increases, the more evident the fluctuations on pressure profile and film profile. The same behavior is shown by Kumar and Kumar [19] and Huaiju Liu [27]. The fluctuation on pressure is attributed to the fact that the surface roughness reduces the fluid flow as the result of the decreased the region available for flow. Therefore, for continuity of flow to be satisfied, higher local pressure will be produced. This means that, increase the roughness amplitude cause greater flow resistance and hence, more pronounced pressure spikes.

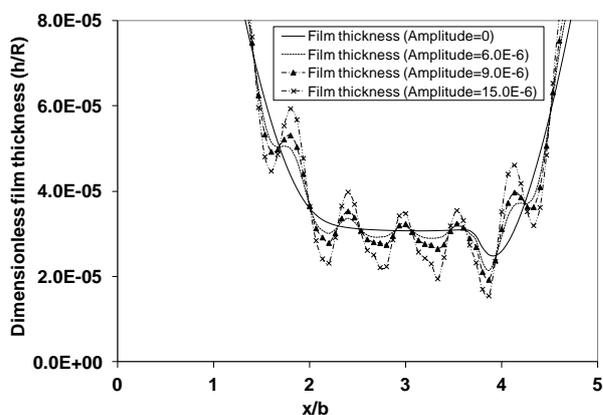


a) Amplitude variation



b) Wavelength variation

Fig. 3. Dimensionless waviness.



a) Dimensionless film profile

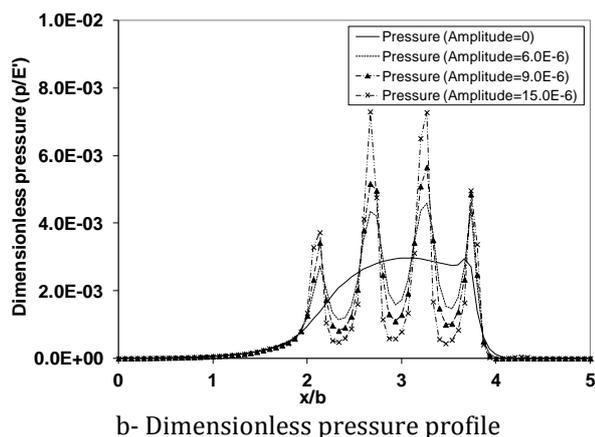


Fig. 4. Dimensionless film profile and pressure for a Newtonian fluid under different amplitude with dimensionless wavelength of 4×10^{-5} .

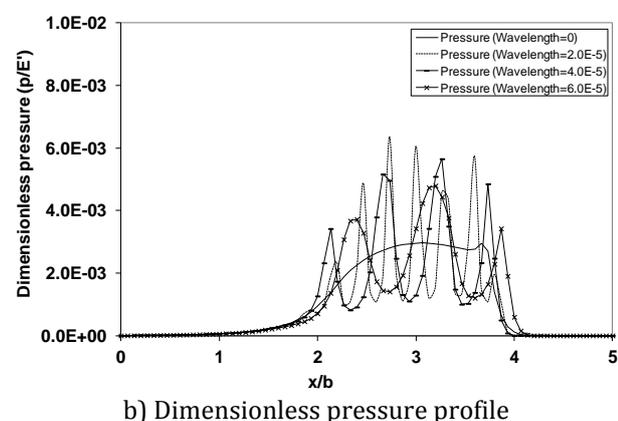
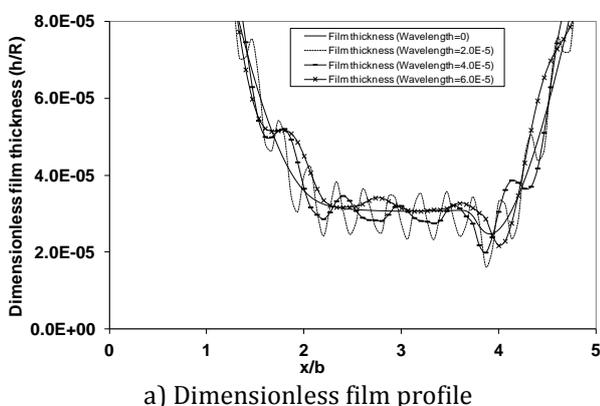


Fig. 5. Dimensionless film profile and pressure for a Newtonian fluid under different wavelength with dimensionless amplitude of 9×10^{-6} .

The effect of changing roughness wavelength on the formation of film shapes and pressure distributions are shown in Figs. 5a and 5b for Newtonian fluid for the same working condition mentioned above with wavelength varied from 2×10^{-5} to 6×10^{-5} and roughness amplitude fixed at 9×10^{-6} . It is clear that, the amplitude of the pressure fluctuation becomes more remarkable as the wavelength of surface roughness

decreases and the minimum film thickness also decrease. This in fact due to the greater flow resistance at sharp asperities (see Kumar and Kumar [19]). Therefore, short wavelength in surface topography must be reduced during the selection of the machining process parameters.

The effect of using non-Newtonian lubricant rheology based on Ree-Eyring model on the behavior of rough surface under different parameters of surface waviness of amplitude and wavelength is shown and discussed in the following part of the paper. In this case, the assumption of Newtonian fluid used in elastohydrodynamic lubrication problem is replaced by a non-Newtonian model to be more realistic and in this case the problem becomes more complicated.

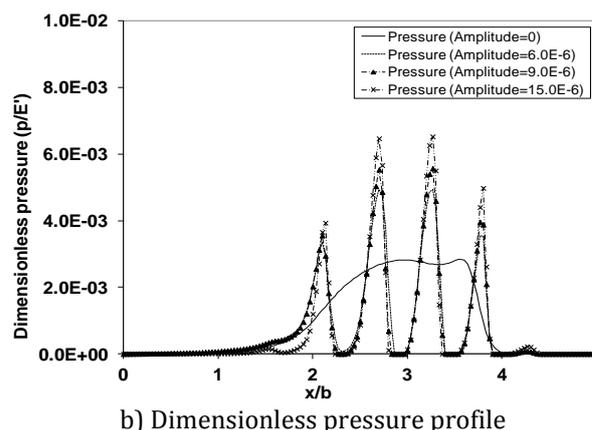
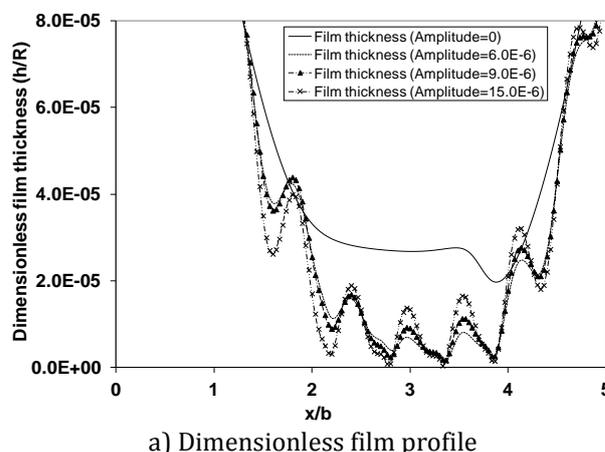


Fig. 6. Dimensionless film profile and pressure for a non-Newtonian fluid under different amplitude with dimensionless wavelength of 4×10^{-5} .

Figures 6a and 6b shows the film thickness and pressure profile obtained numerically using non-Newtonian fluid based on Ree-Eyring model for the same operating conditions used previously for a Newtonian case with amplitude

variation in dimensionless form from 6×10^{-6} to 15×10^{-6} and at fixed dimensionless wavelength of 4×10^{-5} . It is clear that, as the amplitude of surface roughness increase, the fluctuations on pressure profile increase. The non-parallel of film thickness through the contact zone which is the characteristic of the non-Newtonian model for lubrication in addition to the fluctuations in film shape appears. Comparing the non-Newtonian solution with that the Newtonian one for rough surfaces, it is clear that the fluctuation of pressure amplitudes slightly decrease and the film thickness is much thinner in addition to that the film profile is not parallel through the contact zone as compared to that for the Newtonian case (see Fig. 7, for example, for the dimensionless amplitude value of 9×10^{-6} and dimensionless wavelength value of 4×10^{-5}).

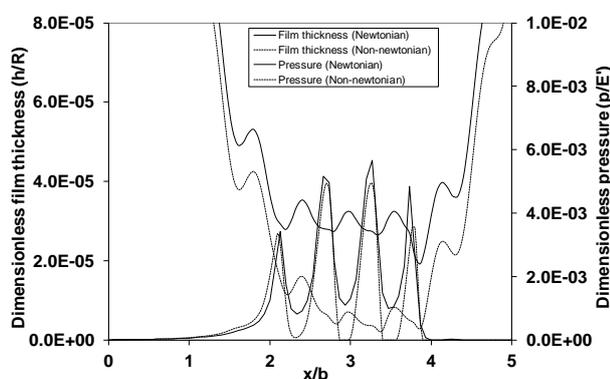
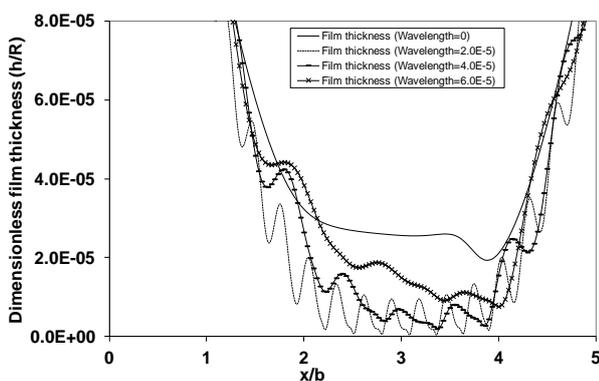
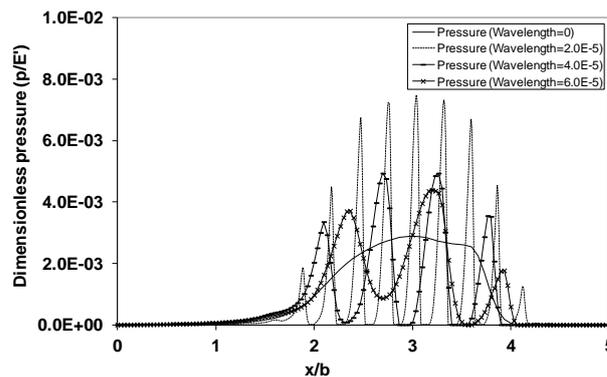


Fig. 7. Dimensionless film profile and pressure for Newtonian and non-Newtonian fluid for values of amplitude of 9×10^{-6} and wavelength of 4×10^{-5} .

Study the effect of changing the wavelength of surface roughness for non-Newtonian lubricant on the formation of film thickness and pressure distribution is shown in Figs. 8a and 8b. In this figure, the wavelength varied from 2×10^{-5} to 6×10^{-5} and the roughness amplitude fixed at 9×10^{-6} for the previously mentioned operating conditions.



a) Dimensionless film profile



b) Dimensionless pressure profile

Fig. 8. Dimensionless pressure and film profile for a non-Newtonian fluid under different wavelength with dimensionless amplitude of 9×10^{-6} .

It is clear that, as the wavelength decreases, the amplitude of the pressure fluctuation becomes more pronounced. The same features as noticed by using the Newtonian model. The film thickness profile shows that a reduction in the film thickness as the wavelength decrease. The film profile is not as parallel as the Newtonian one within the Hertzian contact region in addition to the fluctuation in the film profile as the result of the surface topography. Increasing the wavelength leads to increasing the film thickness and at the same time less fluctuation for film profile.

4. CONCLUSION

The influence of lubricant rheology and sinusoidal waviness for surface roughness on the elastohydrodynamic lubrication of point contact problem has been studied. For this purpose, a numerical solution using surfaces with different roughness wavelength and amplitude for shear-thinning lubricants described by Ree-Eyring model were done using the Newton-Raphson technique with Gauss-Seidel iteration method to solve the Reynolds equation. The results show that, for Newtonian and non-Newtonian model, the pressure and film shapes are functions of amplitude and wavelength for the wavy rough surface. Increasing the amplitude of surface waviness causes the reduction in the film thickness and at the same time increase the random fluctuation of pressure. The effect of wavelength on film thickness and pressure characteristic diminishes with increasing wavelength. The difference between both models of Newtonian and non-

Newtonian lubricant is that the non-parallel nature of the film thickness during the contact zone in addition to the reduction in the film thickness appears for non-Newtonian rheology as a feature for shear-thinning lubricants of Ree-Eyring model. Finally, the fluctuation of pressure amplitudes slightly decrease for non-Newtonian model compared to the Newtonian model.

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NOMENCLATURE

A	: Dimensionless amplitude	u	: Speed of entraining motion, $u = (u_A + u_B)/2$
b	: Radius of Hertzian contact region	U^*	: Speed (or Rolling Viscosity) parameter, $U^* = u\eta_0/ER_x^2$
E	: Reduced modulus of elasticity	Z	: Viscosity-pressure index
G^*	: Materials' parameter, $G^* = E'\alpha$	α	: Pressure of viscosity coefficient
h	: Lubricant film thickness	δ	: Total elastic deformation
H	: Dimensionless film thickness, $H = hR_x/b^2$	ε, ξ	: Constants used in equation (4)
H_0	: Dimensionless central oil film thickness	η	: Lubricant dynamic viscosity
l	: Dimensionless side leakage boundary distance	η_0	: Atmospheric lubricant dynamic viscosity
m	: Dimensionless inlet distance	ρ	: Lubricant density
P	: Hydrodynamic contact pressure	ρ_0	: Atmospheric lubricant density
p	: Dimensionless Hydrodynamic contact pressure, $p = P/P_{Her}$	$\bar{\rho}$: Dimensionless lubricant density, $\bar{\rho} = \rho/\rho_0$
P_{Her}	: Maximum Hertzian contact pressure	$\bar{\eta}$: Dimensionless lubricant viscosity, $\bar{\eta} = \eta/\eta_0$
K	: Elliptical ratio	λ	: Dimensionless wavelength
R_x	: Radius of counterformal contact	τ_0	: Eyring model shear stress
w	: Normal applied contact load	τ_m	: Mean shear stress
W^*	: Load parameter, $W^* = w/ER_x^2$	γ'	: Shear rate
X, Y	: Dimensionless co-ordinates, $X = x/b, Y = y/b$	τ_e	: Equivalent shear stress.